Concept of Sequence :-

Sequence :- A function $f: N \to S$ from the set N of natural numbers into a given set S is called a sequence (of points) in S. For each $n \in N$, the image $a_n = f(n)$ is called n^{th} term of the sequence f.

It is clear that a sequence $f: N \to S$ is a subset of the cartesian product $N \times S$, and may be written as

$$f = \langle \langle 1, f(1) \rangle, \langle 2, f(2) \rangle, \ldots, \langle n, f(n) \rangle, \ldots \rangle$$

It is customarily written as

$$f = \langle f(1), f(2), \ldots, f(n), \ldots \rangle$$

or, $f = \langle a_1, a_2, \ldots, a_n, \ldots \rangle$,

where we define $f(n) = a_n$, for each $n \in N$,

or $f = \langle a_n \rangle$.

For example, $\langle \frac{1}{n} \rangle$ denotes the sequence $\langle 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots \rangle$ with the n^{th} term $\frac{1}{n}$, and $\langle (-1)^n \rangle$ denotes the sequence $\langle -1, 1, -1, 1, \ldots, (-1)^n, \ldots \rangle$. In the latter sequence there are only two distinct elements yet there are infinitely many terms of the sequence. This sequence should not be confused with the set $\{-1, 1\}$, such a set may be called the associated set of the sequence.

A sequence whose associated set is singleton, is called a constant sequence.

Sometimes we write a sequence by a particular law known as a recursion formula. For examples

$a_1 = 1$, $a_{n+1} = \sqrt{(2+a_n)}$, for $n \geq 1$.

which defines a sequence whose terms are

$1, \sqrt{3}, \sqrt{2+\sqrt{3}}, \sqrt{2+\sqrt{2+\sqrt{3}}}, \ldots$

It may be noted that a sequence be an infinite sequence, its range (i.e. the set of function values) need not be infinite.

A sequence whose range is a linear set is called a sequence of real numbers (or real sequence). Similarly, we may define a sequence of complex numbers.

In this unit the term sequence implies a real infinite sequence.

<u>Bounded above sequence</u>:- A sequence $\langle a_n \rangle$ is said to be bounded above if $\exists$ a constant number $M \in R$ s.t.

$$a_n \leq M \quad \forall n \in N.$$

The constant number M is called an upper bound of a_n.

<u>Example</u>:- The sequence $\langle a_n \rangle = \langle \frac{1}{n} \rangle$ is bounded above, because $a_n \leq 1$, $\forall n \in N$

and 1 is upper bound of $\langle \frac{1}{n} \rangle$.

<u>Bounded below sequence</u>:- The sequence $\langle a_n \rangle$ is said to bounded below if $\exists$ a constant number $m \in R$ s.t.

$$m \not> a_n \text{ i.e. } m \leq a_n \quad \forall n \in N.$$

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The constant number m is called lower bound of a_n.

Example: The sequence $\langle a_n\rangle = \langle n\rangle$ is bounded below, because $1 \le a_n$, $\forall n \in \mathbb{N}$

and 1 is lower bound of $\langle n\rangle$.

It is not bounded above.

Bounded Sequence: The sequence $\langle a_n\rangle$ is said to bounded if it is both bounded above and bounded below i.e. $\exists\ m, M \in \mathbb{R}$ such that

$$m \le a_n \le M, \quad \forall n \in \mathbb{N}.$$

Example: The sequence $\langle a_n\rangle = \langle \frac{1}{n}\rangle$ is bounded sequence, since $0 \le \frac{1}{n} \le 1$; $\forall n \in \mathbb{N}$.

Here 0 is lower bound and 1 is upper bound.

We can also say that a sequence $\langle a_n\rangle$ is said to be bounded sequence if there exists a number K s.t. $|a_n| \le K$, $\forall n \in \mathbb{N}$

i.e. $-K \le a_n \le K$, $\forall n \in \mathbb{N}$

Exercise:

(i) The sequence $\langle (-1)^n\rangle$ is bounded sequence.

(ii) The sequence $\langle n\rangle$ is bounded below only and not bounded above and so it is unbounded.

(iii) The sequence $\langle a_n\rangle = \langle -n\rangle$ is bounded above but not bounded below.

(iv) The sequence $\langle a_n\rangle = \langle (-1)^n \sqrt{n}\rangle$ is neither bounded above nor bounded below.

Convergence and limit :- A sequence $\langle a_n \rangle$ is said to converge to a number L. If for any $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$|a_n - L| < \epsilon \text{ for } n > n_0$$

The number L is called the limit of the sequence $\langle a_n \rangle$.

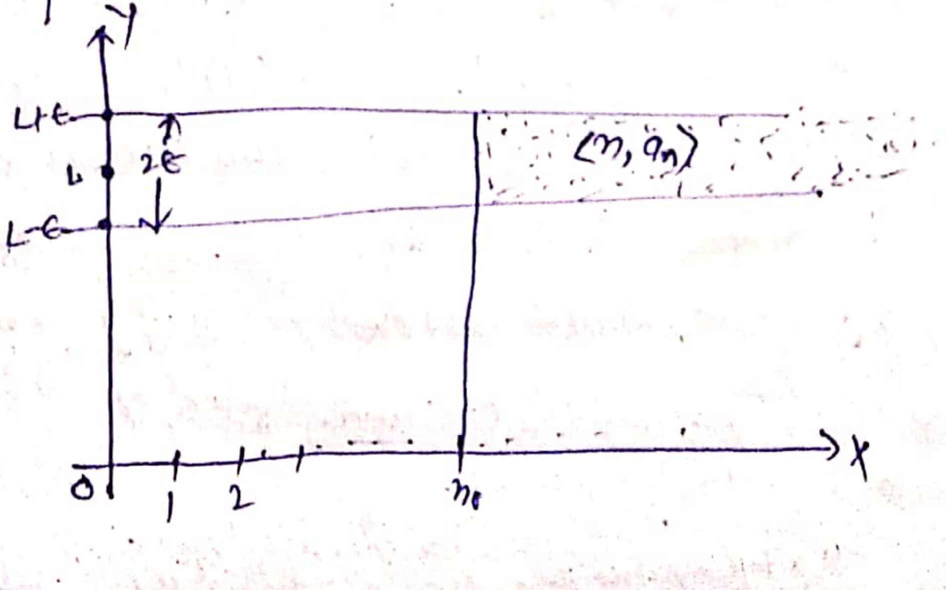
we write $\lim\limits_{n \to \infty} a_n = L$ or $\lim a_n = L$ or $a_n \to L$ as $n \to \infty$ or simply $a_n \to L$, and say that the sequence is convergent and converges to a number L.

If $L = 0$, then $\langle a_n \rangle$ is called a null sequence.

Sequence which are not convergent may be called non-convergent.

Geometrical Significance of $\lim a_n = L$:-

Noticing $f = \{\langle n, a_n \rangle : n \in \mathbb{N}\}$, the graph of f shows that the $\lim a_n = L$ means that how narrow a strip $(y = L \pm \epsilon)$ we take, we can always find a line $x = n_0$, such that all the points $\langle n, a_n \rangle$ to the right of $x = n_0$ lie within this strip.



Theorem: Every convergent sequence is bounded.

Proof: Let $\langle a_n \rangle$ be a convergent sequence and let $a_n \to L$.

For $\epsilon = 1$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$|a_n - L| < 1 \text{ for } n > n_0$$

$$\Rightarrow |a_n| < |L| + 1 \text{ for } n > n_0$$

Let $M = \max \langle |a_1|, |a_2|, \ldots, |a_{n_0}|, |L| + 1 \rangle$

Then $|a_n| \le M$, for $n = 1, 2, 3, \ldots$ which proves the theorem.

This definition is restricted definition for convergence in the sense that it requires L to be finite.

The converse of this theorem is not necessarily true. For instant $\langle (-1)^n \rangle$ is bounded but not convergent.

Theorem: The limit of convergent sequence is unique. or A convergent sequence can have at most one limit.

Proof: Suppose, if possible the convergent sequence $\langle a_n \rangle$ has two limits L & L'. Then $\lim_{n \to \infty} a_n = L \Rightarrow$ for every $\epsilon > 0$, $\exists\, n_1 \in \mathbb{N}$ s.t.

$$|a_n - L| < \epsilon/2 \quad \forall\, n > n_1$$

and $\lim_{n \to \infty} a_n = L' \Rightarrow$ for every $\epsilon > 0$ $\exists\, n_2 \in \mathbb{N}$ s.t.

$$|a_n - L'| < \epsilon/2 \quad \forall\, n > n_2$$

Let $n_0 = \max\{n_1, n_2\}$. Then

$|a_n - L| < \epsilon/2$ and $|a_n - L'| < \epsilon/2 \quad \forall n > n_0$

Now for $n > n_0$

$$\therefore |L - L'| \leq |L - a_n| + |a_n - L'|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

i.e. $|L - L'| < \epsilon$

Since ϵ is arbitrary +ve real number.

$\therefore L = L'$

Hence limit of $\langle a_n \rangle$ is unique.

Divergent sequence: A sequence $\langle a_n \rangle$ is said to diverge to $-\infty$ (minus infinity) iff given any number $k > 0$ (however large) $\exists$ $n_0 \in \mathbb{N}$ such that $-a_n > k$

i.e. $a_n < -k \quad \forall n > n_0$

The above fact written as

$$\lim_{n \to \infty} a_n = -\infty \quad \text{or} \quad \lim a_n = -\infty$$

or $a_n \to -\infty$ as $n \to \infty$

A sequence $\langle a_n \rangle$ is said to diverge to $+\infty$ (plus infinity) iff given any number $k > 0$ (however large) $\exists$ $n_0 \in \mathbb{N}$ s.t. $a_n > k, \forall n > n_0$

In terms of symbols; written as

$$\lim_{n \to \infty} a_n = +\infty \quad \text{or} \quad \lim a_n = +\infty$$

or $a_n \to \infty$ as $n \to \infty$

Exercise :- (i) $\langle n \rangle$ diverges to $+\infty$ (ii) $\langle -n \rangle$ diverges to $-\infty$.
(iii) $\langle 2n - n^2 \rangle$ diverges to $-\infty$.

Oscillatory Sequence: A sequence is said to be an oscillatory sequence if it is neither convergent nor divergent.

An oscillatory sequence is said to oscillate finitely or infinitely according as it is bounded or unbounded.

Examples: (I) $\langle 1+(-1)^n \rangle$ oscillates finitely.

(II) $\langle (-1)^n (1+\frac{1}{n}) \rangle$ oscillates finitely.

(III) $\langle n(-1)^n \rangle$ oscillates infinitely.

Note that a sequence diverge or oscillates infinitely iff its reciprocal sequence exists and converges to 0.

Example: Show that the sequence $\langle \frac{n}{n+1} \rangle$ converges to 1.

Solution: Let ϵ be any +ve number

Then $\left| \frac{n}{n+1} - 1 \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} < \epsilon$ if $n > \frac{1-\epsilon}{\epsilon}$

Let us choose a +ve integer $n_0 > \frac{1-\epsilon}{\epsilon}$

Then for $\epsilon > 0$ $\exists$ $n_0 \in N$ s.t.

$$\left| \frac{n}{n+1} - 1 \right| < \epsilon \quad \forall\, n > n_0$$

Hence $\lim \frac{n}{n+1} = 1$

The sequence $\langle \frac{n}{n+1} \rangle$ is bounded

Since $\frac{1}{2} \le \frac{n}{n+1} < 1$ $\forall\, n \in N$.

Also this sequence being monotonic non-decreasing converges to its lub 1.

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Example: Show that the sequence $\langle a_n = \frac{1}{r^n} \rangle$ converges to zero if $|r|<1$.

Solution: We set

$$|r| = \frac{1}{1+h}, h>0 \quad (\because |r|<1)$$

Then $|a_n - 0| = |r^n| = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh}, \forall n$

because $h>0$

and $(1+h)^n = 1+nh + \frac{n(n-1)}{2}h^2 + \cdots + h^n$
$\geq 1+nh, \forall n$

Consequently for given $\epsilon>0$,

$$|a_n - 0| < \epsilon \text{ if } n > \frac{1-\epsilon}{\epsilon h}$$

Let us choose a +ve integer $n_0 > \frac{1-\epsilon}{\epsilon h}$

Then $|a_n - 0| < \epsilon, \quad \forall n > n_0$

Hence $\lim a_n = 0$.

Example: Find the least +ve integer n so that $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$ whenever $n > n_0$.

Solution: The given inequality implies $\frac{6}{n+3} < \frac{1}{5}$ so that $n>27$

Let us choose a +ve integer $n_0 > 27$ such that

$$\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5} ; \forall n > n_0$$

Hence for $\epsilon = \frac{1}{5}$, $n_0 = 27$

Moreover, $\lim \frac{2n}{n+3} = 2$

Some Properties of convergent sequence:—

Theorem:— If $\lim a_n = L\ (\neq 0)$, then $\exists$ a +ve number M and a +ve integer no, such that $|a_n| > M$ for $n > n_0$.

Proof:— Let $a_n \to L$. For $\epsilon = \frac{|L|}{2} > 0$ (It exists since $L \neq 0$) $\exists\ n_0 \in N$ such that

$$|a_n - L| < \epsilon \text{ for } n > n_0$$

Now $|L| \leq |L - a_n| + |a_n| < \epsilon + |a_n|$ for $n > n_0$

Where $|a_n| > |L| - \epsilon = \frac{|L|}{2}$

and the proof is complete. Here $M = \frac{|L|}{2}$.

Theorems: (i) If $a_n \to L$, then $|a_n| \to |L|$

(ii) If $a_n \to L_1$, $b_n \to L_2$ and $a_n < b_n\ \forall n$, then $L_1 \leq L_2$

(iii) If $a_n \to L_1$, $b_n \to L_2$, then $(a_n \pm b_n) \to L_1 \pm L_2$

(iv) If $a_n \to L_1$, $b_n \to L_2$, then $\lim a_n b_n = L_1 \cdot L_2$

(v) If $a_n \to L_1$, $b_n \to L_2$ and $b_n \neq 0$ for any n, & $L_2 \neq 0$ then $\frac{a_n}{b_n} \to \frac{L_1}{L_2}$

Example:— Evaluate the limit of a_n, when $a_{n+1} = \frac{4}{2+a_n}$, assuming that $\langle a_n \rangle$ is convergent sequence of +ve numbers.

Solution:— If $\lim a_n = L$, then $\lim a_{n+1} = \lim\limits_{n \to \infty} \left(\frac{4}{2+a_n}\right)$

$$\Rightarrow L = \frac{4}{2+L} \quad \text{i.e.}\quad L^2 + 2L - 4 = 0$$

so that $L = -1 \pm \sqrt{5}$

As $a_n \geq 0$, so $L \geq 0$ and therefore $L = \sqrt{5} - 1$

Limit point of a sequence:– A number l is said to be a limit point of a sequence $\langle a_n \rangle$ if for any $\epsilon > 0$, $a_n \in (l-\epsilon, l+\epsilon)$, for infinitely many values of $n \in \mathbb{N}$.

Example:– (i) $\langle \frac{1}{n} \rangle$ has only one limit point, i.e. 0 –

(ii) $\langle n \rangle$ has no limit point.

(iii) $\langle (-1)^n \rangle$ has 1 and -1 as its only limit points, whereas its range $\{1, -1\}$ has no limit point.

Bolzano – Weierstrass Theorem for sequences:- Every bounded sequence has at least one limit point.

Proof:– Let S be the associated set of the bounded sequence $\langle a_n \rangle$. If S is finite, then at least one term, say $a_l = l$, must occur denumerably many times in $\langle a_n \rangle$, and hence l is a limit point of $\langle a_n \rangle$. If S is infinite, then, being bounded, it must have a limit point, say l. Consequently, there is a sequence of distinct points of S whose limit is l. Since S is the range set of $\langle a_n \rangle$, it follows that l is a limit point of $\langle a_n \rangle$. Hence in either case $\langle a_n \rangle$ has at least one limit point, which completes the proof.

The converse of this theorem is not true necessarily. For example, $\langle 1, 3, 1, 5, 1, 7, \ldots \rangle$ has a unique limit point but is unbounded above.

Cauchy Sequence or Fundamental Sequence :-

A sequence $\langle a_n \rangle$ is said to be a cauchy-sequence if for any $\epsilon > 0$, $\exists$ $n_0 = n_0(\epsilon) \in N$ s.t.

$$|a_n - a_m| < \epsilon \quad \forall\, n, m \geq n_0$$

or A sequence $\langle a_n \rangle$ is said to be a Cauchy-sequence, if for any given $\epsilon > 0$, $\exists$ a +ve integer $n_0(\epsilon)$ such that

$$|a_{n+p} - a_n| < \epsilon \quad \forall\, n \geq n_0 \text{ and } p \geq 1.$$

Example :- The sequence $\langle \frac{1}{n} \rangle$ is a Cauchy-sequence.

Solution :- For, let $\epsilon > 0$ be given, Then

$$|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| < \frac{1}{m} \quad \text{if } n \geq m.$$

If we choose $n_0 \in N$ s.t. $n_0 > \frac{1}{\epsilon}$ i.e. $\frac{1}{n_0} < \epsilon$,

then we have $|a_n - a_m| < \frac{1}{m} \leq \frac{1}{n_0} < \epsilon$

$\forall\, n, m \geq n_0$

Hence $\langle \frac{1}{n} \rangle$ is a Cauchy sequence.

Theorem :- Every Convergent sequence is a Cauchy-sequence.

Proof :- Let $\langle a_n \rangle$ be a convergent sequence and it converge to the limit L. Then for any given $\epsilon > 0$, $\exists$ a +ve integer $n_0(\epsilon)$ s.t.

$$|a_n - L| < \epsilon/2 \; ; \; \forall\, n \geq n_0$$

& also $|a_m - L| < \epsilon/2$, for $m \geq n_0$

Now $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L| < \epsilon/2 + \epsilon/2$

$\Rightarrow |a_n - a_m| < \epsilon \quad \forall\, n, m \geq n_0$

Hence $\langle a_n \rangle$ is a Cauchy sequence.

Theorem: Every Cauchy sequence is bounded.

Proof: Let $\langle a_n \rangle$ be a Cauchy sequence, then for $\epsilon = 1$, $\exists$ $n_0 \in N$ s.t. $|a_n - a_m| < 1$, for $n, m \geq n_0$

In particular, $|a_n - a_{n_0}| < 1$

$$\Rightarrow |a_n| < |a_{n_0}| + 1$$

Let $M = \max.\{|a_1|, |a_2|, \ldots, |a_{n_0 - 1}|, |a_{n_0}+1|\}$

Then $|a_n| \leq M$; $\forall n \in N$

Hence $\langle a_n \rangle$ is bounded.

But the converse of this theorem is not necessarily true. For instance, $\langle (-1)^n \rangle$ is bounded sequence because $|(-1)^n| = 1$ $\forall n \in N$, which it is not a cauchy-sequence.

As $|a_{n+1} - a_n| = 2$; for any $n \in N$.

Taking $\epsilon = 1$; we infer that

$$|a_{n+1} - a_n| \nless \epsilon \text{ for any } n \in N.$$

Theorem: A Cauchy sequence has a unique limit point.

Proof: Cauchy sequence $\langle a_n \rangle$ being bounded must have a limit point, say l. We claim that l is the only limit point of $\langle a_n \rangle$.

If l' is another limit point of $\langle a_n \rangle$, then by setting $\epsilon = \frac{1}{2}|l - l'|$; we have disjoint ϵ-nbds of l and l' and both of them contain infinitely many terms of the sequence $\langle a_n \rangle$, which is impossible., Hence $l = l'$, and the proof is complete.